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1. Introduction.

Let H be the space of n-vector valued functions[†] $y(\varphi) = (y_1(\varphi),\ldots,y_n(\varphi))'$ on the real finite interval $[-r,0]$, $r > 0$, whose components are continuous on $[-r,0]$. Suppose $x(t)$ is an n-vector valued function defined on the real interval $[-r,T]$, $T > 0$. Fix $t \in [0,T]$. Let x_t denote the element of H with values $x(t+\varphi)$ at φ, $\varphi \in [-r,0]$. Let $x(\cdot)$ be the solution of the delay equation[††]

$$(1)\qquad \dot{x}(t) = A(t)x(t) + B(t)x(t-r) + \int_{-r}^{0} C(t,\varphi)x(t+\varphi)d\varphi + D(t)u(t)$$

where $A(t)$, $B(t)$, $C(t,\varphi)$, $D(t)$, and the derivatives of $B(t)$ and $C(t,\varphi)$ for $(t,\varphi) \in [0,T] \times [-r,0]$, and the 'initial condition', x_0, is in H.

This paper is concerned with finding the control $u(\cdot)$ which minimizes the quadratic functional

$$(2)\qquad V^u(x_t,t) = \int_t^T [x'(s)M(s)x(s) + u'(s)N(s)u(s)]ds,$$

where $M(s)$ and $N(s)$ are continuous[†††], $M(s) \geq 0$, and $N(s) > 0$ for

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[†] The prime ' denotes transpose.

[††] (1) is treated for simplicity; it will be obvious that replacing the term $Bx(t-r)$ by $\Sigma B_i x(t-r_i)$ demands few changes in the development.

[†††] $M \geq 0$, $N > 0$ denote that M is non-negative definite and N is positive definite.

each s in $[0,T]$. Special forms have been considered by other authors, e.g. Krasovskii [1]; however, that work is quite vague and, in particular, the crucial fact that the relevant 'Ricatti-like' equation has a solution of the proper form or even some solution is not shown. Since the 'Ricatti' equation is a rather complicated coupled set of first order partial differential equations, this question requires some treatment. Theorems 1 and 2 give the representation of $V(x_t,t)$ as a quadratic functional of x_t, Theorem 3 proves the smoothness of solutions to certain partial differential equations, and Theorems 4 and 5 contain the basic result on iteration in policy space. Theorem 6 is the final optimization theorem. Unfortunately, as is common with works on functional-differential equations, some of the calculations are somewhat tedious. Although the problem has an intrinsic interest of its own, owing to the appearance of delays in many situations, the authors interest in it stemmed from an attempt to analyze a problem where $u(t)$ was actually a functional of noise corrupted observations taken on the interval $[t-r,t]$. This was part of an attempt to use the theory of stochastic delay equations to study certain approximations to non-linear filters, and to stabilize a system when only noise corrupted observations are available. The latter investigation led to the consideration of the problem of the paper. See Barnea [2].

2. <u>A Preliminary Lemma.</u>

<u>Lemma 1.</u> <u>Let</u> $u = 0$ <u>and let the</u> $A(t)$, $B(t)$, $\partial B(t)/\partial t$, $\partial C(t,\varphi)/\partial t$ <u>and</u> $C(t,\varphi)$ <u>be continuous. Then the solution</u> $x(s)$ <u>has the representation, for</u> $s \geq t$,

$$(3) \qquad x(s) = K(s,t)x(t) + \int_{-r}^{0} \tilde{K}(s,t,\varphi)x(t+\varphi)d\varphi$$

where $K(s,t) = 0$ *for* $s < t$, $K(t,t) = I$, *the identity, and* $K(s,t)$ *is continuous in* (s,t) *for* $s \geq t$. *For fixed* t, *it satisfies* (1), *as a function of* s (*with* $u = 0$). *For fixed* s, *it satisfies* (*as a function of* t) *the adjoint of* (1) (*with* $u = 0$), *for* $t \leq s$. *The terms* $\partial K(s,t)/\partial s$ *and* $\partial K(s,t)/\partial t$ *are continuous for* $s \geq t$ *except for a finite discontinuity at* $s = t + r$. *Also*

$$(4) \quad \tilde{K}(s,t,\varphi) = K(s,t+r+\varphi)B(t+r+\varphi) + \int_{-\varphi}^{r} K(s,t+\varphi+\rho)C(t+\varphi+\rho,-\rho)d\rho.$$

(*The upper limit* r *can be replaced by* $\min(s-t-\varphi,r)$.) *The first term on the right of* (4) *is zero for* $s < t + r + \varphi$, *continuous in* (s,t,φ) *for* $s \geq t + r + \varphi$, *and its derivatives with respect to* s,t,φ *are continuous for* $s \geq t + r + \varphi$, *except at* $s = t + 2r + \varphi$, *where there is a finite discontinuity. The second term of* (4) *is zero for* $s < t$ *and is continuous together with its derivatives with respect to*[†] s,t,φ *for* $T \geq s \geq t \geq 0$, $-r \leq \varphi \leq 0$.

Note. $\tilde{K}(s,t,\varphi) = 0$ for $s < t$. For the computations of Theorem 1, it is convenient to redefine $\tilde{K}(s,t,\varphi)$ for $s < t$ so that (3) gives the solution for $s \geq t - r$. Then define $\hat{K}(s,t,\varphi) = \tilde{K}(s,t,\varphi)$ for $s \geq t$ and, for $t - r \leq s < t$, define the symbol $\int_{-r}^{0} \hat{K}(s,t,\varphi)x(t+\varphi)d\varphi$ to mean

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[†]By convention, if $s = t + r + \varphi$, the derivative with respect to s is a right-hand derivative, and with respect to t and φ a left-hand derivative; i.e., the limits are taken within the segment $s \geq t + r + \varphi$.

$x(s)$; i.e., for $s < t$, $\hat{K}(s,t,\varphi)$ is the Dirac δ-function $\delta(s-(t+\varphi))$. Thus for $s \geq t - r$,

$$(3') \qquad x(s) = K(s,t)x(t) + \int_{-r}^{0} \hat{K}(s,t,\varphi)x(t+\varphi)d\varphi.$$

Proof. The forms (3), (4) and statements concerning $K(s,t)$ follow from Halanay [3], p. 369-370. The statements concerning $\tilde{K}(s,t,\varphi)$ are straightforward consequences of the properties $K(s,t)$, by virtue of the representation (4).

Remark. In (1) let $u(t)$ take the form

$$(5) \qquad u(t) = E_u(t)x(t) + \int_{-r}^{0} F_u(t,\varphi)x(t+\varphi)d\varphi$$

Then

$$(1') \qquad \dot{x}(t) = A_u(t)x(t) + B(t)x(t-r) + \int_{-r}^{0} C_u(t,\varphi)x(t+\varphi)$$

where

$$A_u(t) = A(t) + D(t)E_u(t)$$

$$C_u(t,\varphi) = C(t,\varphi) + D(t)F_u(t,\varphi).$$

Let $D(t)$, $E_u(t)$, $F_u(t,\varphi)$, $\partial D(t)\partial t$ and $\partial F_u(t,\varphi)/\partial t$ be continuous. Then, Lemma 1 remains valid, where we replace $K,\tilde{K}$ by $K_u,\tilde{K}_u$, resp., the kernels corresponding to (1').

3. Representations for the Cost.

By substituting (5) into (2), we obtain

$$
\begin{aligned}
V^u(x_t,t) &= \int_t^T (x'(s)M_u(s)x(s))ds \\
&+ \int_t^T ds(\int_{-r}^{0} d\varphi x'(s)L_u(s,\varphi)x(s+\varphi)) + \int_t^T ds(\int_{-r}^{0} d\varphi x'(s+\varphi)L_u'(s,\varphi)x(s)) \\
&+ \int_t^T ds(\int_{-r}^{0} d\varphi \int_{-r}^{0} d\rho x'(s+\varphi)G_u(s,\varphi,\rho)x(s+\rho)) \\
&\equiv T_1 + T_2 + T_3 + T_4
\end{aligned} \tag{6}
$$

where the T_i are the terms on the right of (6), and

$$
\begin{aligned}
M_u(s) &= M(s) + E_u'(s)N(s)E_u(s) \\
L_u(s,\varphi) &= E_u'(s)N(s)F_u(s,\varphi) \\
G_u(s,\varphi,\rho) &= F_u'(s,\varphi)N(s)F_u(s,\rho).
\end{aligned} \tag{7}
$$

Theorem 1. Let $u(t)$ take the form (5), and assume the conditions of Lemma 1 and the remark following it. In addition, let $\partial C(t,\varphi)/\partial\varphi$ and $\partial F_u(t,\varphi)/\partial\varphi$ be continuous and $F_u(t,\varphi)$ and $E_u(t)$ tend to zero as $t \to T$. Let $M(s)$ and $N(s)$ be symmetric and continuously differentiable for $s \in [0,T]$. Then[†,††]

[†] The $S_i, \hat{S}_i$ are defined as the terms on the right of (8).

[††] If (2) contains a terminal cost term $x'(T)Zx(T)$, then (9), (10), (11) would each contain one additional term (which is not of an integral form). However, we have not been able to show that the additional terms have the smoothness that we will require (i.e. be differentiable).

$$V^u(x_t,t) = S_1 + S_2 + \tilde{S}_2 + S_3$$

$$= x'(t)P_u(t)x(t) + x'(t)\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi$$

(8) $$+ \int_{-r}^{0} x'(t+\varphi)Q_u'(t,\varphi)x(t)d\varphi$$

$$+ \int_{-r}^{0} d\varphi \int_{-r}^{0} d\rho x'(t+\varphi)R_u(t,\varphi,\rho)x(t+\rho).$$

<u>The</u> $P_u(t)$, $Q_u(t,\varphi)$, $R_u(t,\varphi,\rho)$ <u>are sums of the terms in</u> (9), (10), (11), <u>resp.</u>

(9a) $$P_{u1}(t) = \int_t^T K_u'(s,t)M_u(s)K_u(s,t)ds$$

(9b) $$P_{u2}(t) = \int_t^T ds \int_{-r}^{0} d\tau K_u'(s,t)L_u(s,\tau)K_u(s+\tau,t)$$

(9c) $$P_{u3}(t) = P_{u2}'(t)$$

(9d) $$P_{u4}(t) = \int_t^T ds \int_{-r}^{0} d\varphi \int_{-r}^{0} d\rho K_u'(s+\varphi,t)G_u(s,\varphi,\rho)K_u(s+\rho,t)$$

(10a) $$Q_{u1}(t,\varphi) = \int_t^T K_u'(s,t)M_u(s)\hat{K}_u(s,t,\varphi)ds = \int_t^T K_u'(s,t)M_u(s)\tilde{K}_u(s,t,\varphi)$$

$$Q_{u2}(t,\varphi) = \int_t^T ds \int_{-r}^{0} d\tau K_u'(s,t)L_u(s,\tau)\hat{K}_u(s+\tau,t,\varphi)$$

(10b) $$= \int_t^{\min[t+r+\varphi,T]} ds K_u'(s,t)L_u(s,t-s+\varphi)$$

$$+ \int_t^T ds \int_{-r}^{0} d\tau K_u'(s,\tau)L_u(s,\tau)\tilde{K}_u(s+\tau,t,\varphi)$$

$$Q_{u3}(t,\varphi) = \int_t^T ds \int_{-r}^{0} d\tau K_u'(s+\tau,t)L_u'(s,\tau)\hat{K}_u(s,t,\varphi)$$

(10c) $$= \int_t^T ds \int_{-r}^{0} d\tau K_u'(s+\tau,t)L_u'(s,\tau)\tilde{K}_u(s,t,\varphi)$$

$$Q_{u4}(t,\varphi) = \int_t^T ds \int_{-r}^0 d\alpha \int_{-r}^0 d\rho K_u'(s+\alpha,t)G_u(s,\alpha,\rho)\hat{K}_u(s+\rho,t,\varphi)$$

$$\text{(10d)} \qquad = \int_t^{\min[t+r+\varphi,T]} ds \int_{-r}^0 d\alpha K_u'(s+\alpha,t)G_u(s,\alpha,t-s+\varphi)$$

$$+ \int_t^T ds \int_{-r}^0 d\alpha \int_{-r}^0 d\rho K_u'(s+\alpha,t)G_u(s,\alpha,\rho)\tilde{K}_u(s+\rho,t,\varphi)$$

$$\text{(11a)} \qquad R_{u1}(t,\varphi,\rho) = \int_t^T \hat{K}_u'(s,t,\varphi)M_u(s)\hat{K}_u(s,t,\rho)ds = \int_t^T \tilde{K}_u'(s,t,\varphi)M_u(s)\tilde{K}_u(s,t,\rho)ds$$

$$R_{u2}(t,\varphi,\rho) = \int_t^T ds \int_{-r}^0 d\tau \hat{K}_u'(s,t,\varphi)L_u(s,\tau)\hat{K}_u(s+\tau,t,\rho)$$

$$\text{(11b)} \qquad = \int_t^{\min[t+r+\rho,T]} ds\tilde{K}_u'(s,t,\varphi)L_u(s,t-s+\rho)$$

$$+ \int_t^T ds \int_{-r}^0 d\tau \tilde{K}_u'(s,t,\varphi)L_u(s,\tau)\tilde{K}_u(s+\tau,t,\rho)$$

$$\text{(11c)} \qquad R_{u3}(t,\varphi,\rho) = R_{u2}'(t,\rho,\varphi)$$

$$R_{u4}(t,\varphi,\rho) = \int_t^T ds \int_{-r}^0 d\alpha \int_{-r}^0 d\beta \hat{K}_u'(s+\alpha,t,\varphi)G_u(s,\alpha,\beta)\hat{K}_u(s+\beta,t,\rho)$$

$$= \int_t^{\min[t+r+\varphi,t+r+\rho,T]} G_u(s,t-s+\varphi,t-s+\rho)ds +$$

$$\text{(11d)} \qquad \int_t^{\min[t+r+\rho,T]} ds \int_{-r}^0 d\alpha \tilde{K}_u'(s+\alpha,t,\varphi)G_u(s,\alpha,t-s+\rho)$$

$$+ \int_t^{\min[t+r+\varphi,T]} ds \int_{-r}^0 d\alpha G_u(s,\alpha,t-s+\varphi)\tilde{K}(s+\alpha,t,\rho)$$

$$+ \int_t^T ds \int_{-r}^0 d\alpha \int_{-r}^0 d\beta \tilde{K}_u'(s+\alpha,t,\varphi)G_u(s,\alpha,\beta)\tilde{K}_u(s+\beta,t,\rho)$$

Furthermore, the T_i have the form (8) where P_u, Q_u and R_u are replaced by P_{ui}, Q_{ui} and R_{ui}, resp. $P_u, Q_u,$ and R_u have bounded derivatives in their arguments for† $0 \le t \le T$, $-r \le \varphi \le 0$, $-r \le \rho \le 0$,

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†At $\varphi = 0$ or $\varphi = r$ or $\rho = 0$ or $\rho = r$ or $t = 0$, the derivatives are replaced by the appropriate one sided derivatives.

and satisfy (12). The derivatives are continuous, except for the φ or ρ derivative of $R_u(t,\varphi,\rho)$ at $\varphi = \rho$ where there may be a finite discontinuity[†].

(12a) $$P_u(T) = Q_u(T,\varphi) = R_u(T,\varphi,\rho) = 0$$

(12b) $$\frac{dP_u(t)}{dt} + A_u'(t)P_u(t) + P_u(t)A_u(t) + Q_u(t,0) + Q_u'(t,0) = - M(t) - E_u'(t)N(t)E_u(t) = -M_u(t)$$

(12c) $$2P_u(t)C_u(t,\varphi) + A_u'(t)Q_u(t,\varphi) + Q_u'(t,\varphi)A_u(t) + 2\frac{\partial Q_u(t,\varphi)}{\partial t} - 2\frac{\partial Q_u(t,\varphi)}{\partial \rho} + R_u(t,\varphi,0) + R_u(t,0,\varphi) = -2E_u'(t)N(t)F_u(t,\varphi) = -2L_u(t,\varphi)$$

(12d) $$C_u'(t,\varphi)Q_u(t,\rho) + Q_u'(t,\varphi)C_u(t,\rho) + \frac{\partial R_u}{\partial t}(t,\varphi,\rho) - \frac{\partial R_u}{\partial \rho}(t,\varphi,\rho) - \frac{\partial R_u}{\partial \rho}(t,\varphi,\rho) = -F_u'(t,\varphi)N(t)F_u(t,\rho) = -G_u(t,\varphi,\rho)$$

(12e) $$B'(t)P_u(t) - Q_u(t,-r) = 0$$

$$B'(t)Q_u(t,\varphi) - R_u(t,-r,\varphi) - R_u'(t,\varphi,-r) + Q_u'(t,\varphi)B(t) = 0$$

Finally, the solution $P_u(t)$, $Q_u(t,\varphi)$, $R_u(t,\varphi,\rho)$ is unique within the class of symmetric[††] differentiable $P_u(t)$, $R_u(t,\varphi,\rho)$ and

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[†] For future reference, we note that the discontinuity in R_u is in the terms R_{u2} and R_{u3}. However, it is easy to verify that R_{u2} and R_{u3} are differentiable in the $(1,-1,-1)$ direction in the (t,φ,ρ) set $[0,T] \times [-r,0]^2$.

[††] By symmetric M we mean $M'(t) = M(t)$; by symmetric $G(t,\rho,\varphi)$, we mean $G(t,\varphi,\rho) = G'(t,\rho,\varphi)$.

differentiable $Q_u(t,\varphi)$.

Proof. The evaluation of the T_i-terms on the right of (6) is straightforward by merely substituting the expressions for $x(s)$, $x(s+\varphi)$ and $x(s+\rho)$ from (3) into the T_i and separating the result into a sum of the form of the right side of (8), where the P_{ui}, Q_{ui}, and R_{ui} are given by (9) - (11). The right sides of (9) - (11) are obtained from the center expressions by replacing $\hat{K}$ by its definition in terms of $\tilde{K}$ and the δ-function, and noting that $\tilde{K}(s,t,\varphi) = 0$ for $s < t$. Then (8) follows by merely summing the T_i. The statement concerning the continuity of the derivatives of P_u, Q_u and R_u follow from Theorem 3 and the differentiability of $M_u(s)$, $L_u(s,\varphi)$ and $G_u(s,\varphi,\rho)$ for $0 \le s \le T$, $-r \le \varphi \le 0$, $-r \le \rho \le 0$.

Now, we evaluate

$$\begin{aligned}\frac{d}{dt}[x'(t)P_u(t)x(t)] &= [A_u(t)x(t)+B(t)x(t-r) + \int_{-r}^{0} C_u(t,\varphi)x(t+\varphi)d\varphi]P_u(t)x(t)\\ &+ x'(t)(\frac{dP_u(t)}{dt})x(t) + x'(t)P_u(t)[A_u(t)x(t)+B(t)x(t-r)\\ &+ \int_{-r}^{0} C_u(t,\varphi)x(t+\varphi)d\varphi]\end{aligned} \tag{13a}$$

$$\begin{aligned}\frac{d}{dt}[x'(t)\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi] &= \frac{d}{dt}[x'(t)\int_{t-r}^{t} Q_u(t,\tau-t)x(\tau)d\tau]\\ &= [A_u(t)x(t)+B(t)x(t-r) + \int_{-r}^{0} C_u(t,\varphi)x(t+\varphi)]'\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi\\ &+ x'(t)[Q_u(t,0)x(t) - Q_u(t,-r)x(t-r)\\ &+ \int_{t-r}^{t} \frac{\partial Q_u(t,\tau-t)}{\partial t}x(\tau)d\tau]\end{aligned} \tag{13b}$$

where

$$(13c) \qquad \int_{t-r}^{t} \frac{\partial Q_u(t,\tau-t)}{\partial t} x(\tau)d\tau = \int_{-r}^{0}\left[\frac{\partial Q_u(t,\varphi)}{\partial t} - \frac{\partial Q_u(t,\varphi)}{\partial \varphi}\right] x(t+\varphi)d\varphi .$$

Similarly,

$$\frac{d}{dt}\left[\int_{-r}^{0}\int_{-r}^{0} d\varphi d\rho x'(t+\varphi)R_u(t,\varphi,\rho)x(t+\rho)\right] = \frac{d}{dt}\int_{t-r}^{t}\int_{t-r}^{t} d\tau d\sigma x'(\tau)R_u(t,\rho-t,\sigma-t)x(\sigma)$$

$$\begin{aligned}
(13d) \qquad &= \int_{t-r}^{t} d\sigma[x(t)'R_u(t,0,\sigma-t) - x'(t-r)R_u(t,-r,\sigma-t)]x(\sigma) \\
&+ \int_{t-r}^{t} d\tau x'(\tau)[R_u(t,\tau,0)x(t) - R_u(t,\tau,-r)x(t-r)] \\
&+ \int_{t-r}^{t}\int_{t-r}^{t} d\tau d\sigma x'(\tau)[\frac{\partial}{\partial t}R_u(t,\tau-t,\sigma-t)x(\sigma) \\
&= \int_{-r}^{0} d\rho[x'(t)R_u(t,0,\rho) - x'(t-r)R_u(t,-r,\rho)]x(t+\rho) \\
&+ \int_{-r}^{0} d\varphi x'(t+\varphi)[R_u(t,\varphi,0)x(t) - R_u(t,\varphi,-r)x(t-r)] \\
&+ \int_{-r}^{0}\int_{-r}^{0} x'(t+\varphi)[\frac{\partial}{\partial t} - \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \rho}]R_u(t,\varphi,\rho)x(t+\rho)d\varphi d\rho .
\end{aligned}$$

Note (for reference in Theorems 5, 6), that the representations (13b), (13c), (13d) are valid if $Q_u(t,\varphi)$ only has a uniformly bounded derivative almost everywhere along each line in the $(1,-1)$ direction in the set $\varphi \in [-r,0]$, $t \in [0,T]$, and if $R_u(t,\varphi,\rho)$ has only a uniformly bounded derivative almost everywhere along each line in the $(1,-1,-1)$ direction in the set $t \in [0,T]$, $\varphi, \rho \in [-r,0]$. These conditions and the differentiability of $P_u(t)$ assure the differentiability (in t) of $V^u(x_t,t)$.

Next, adding (13a), twice (13b) and (13d), and using the substitution (13c), yields an expression for $\partial V^u(x_t,t)/\partial t$. However, $\partial V^u(x_t,t)/\partial t$ also equals the negative of the sum of the bracketed integrands in (6), evaluated at $s = t$. The equality of these two forms† of $\partial V^u(x_t,t)/\partial t$ for all $x_t \in H$ and $0 \leq t \leq T$, implies that the coefficients of like terms in $x(t)$, $x(t+\varphi)$, etc., in each form must be equal. This yields (12). Note that, by <u>construction</u> and Theorem 3, (12) has a smooth symmetric solution; i.e., the terms have continuous derivatives and $P_u(s) = P_u'(s)$, $R_u(t,\varphi,\rho) = R_u(t,\rho,\varphi)$ (except that the φ,ρ derivatives of R_u are discontinuous at $\varphi = \rho$).

Let $\hat{P}(t)$, $\hat{Q}(t,\varphi)$, $\hat{R}(t,\varphi,\rho)$ be differentiable solutions†† to (12) with $\hat{P}(t)$, $\hat{R}(t,\varphi,\rho)$ symmetric and define $Z(x_t,t)$ by (14). Then, by reversing the argument leading to (12), we get $d/dt[Z(x_t,t)] = -x'(t)M(t)x(t) - u'(t)N(t)u(t)$.

$$[x'(t)\hat{P}(t)x(t) + x'(t)\int_{-r}^{0} \hat{Q}(t,\varphi)x(t+\varphi)d\varphi + \int_{-r}^{0} x'(t+\varphi)\hat{Q}'(t,\varphi)x(t)d\varphi + \int_{-r}^{0} d\varphi \int_{-r}^{0} d\rho x'(t+\varphi)\hat{R}(t,\varphi,\rho)x(t+\varphi) = Z(x_t,t). \tag{14}$$

However,

$$Z(X_T,T) = V^u(X_T,T) = 0$$

and

---

† Note that $\partial V^u(x_t,t)/\partial t$ also equals $-x'(t)M(t)x(t) - u'(t)N(t)u(t)$.

†† In fact, it is readily verified that we only need that $\hat{Q}(t,\varphi)$ and $\hat{R}(t,\varphi,\rho)$ have uniformly bounded derivatives a.e., in the $(1,-1)$ and $(1,-1,-1)$ directions on the sets $t \in [0,T]$, $\varphi \in [-r,0]$ and $t \in [0,T]$, $\varphi,\rho \in [-r,0]$, resp. More generally, for uniqueness we only need that $\partial R_u(t,\varphi-t,\rho-t)/\partial t$ and $\partial Q_u(t,\varphi-t)/\partial t$ be uniformly bounded for almost all φ,ρ.

$$Z(x_t,t) - Z(X_T,T) = \int_t^T [x'(s)M(s)x(s) + u'(s)N(s)u(s)]ds$$
$$= V^u(x_t,t) - V^u(X_T,T)$$

or, equivalently

$$Z(x_t,t) = V^u(x_t,t). \tag{15}$$

Using the identity (15), the representations (14) and (8), and the continuity of the $P, \hat{P}, Q, \hat{Q}, R, \hat{R}$, and symmetry of $P, \hat{P}$ and $R, \hat{R}$, it is easily shown that† $P_u(t) = \hat{P}(t)$, $Q_u(t,\varphi) = \hat{Q}(t,\varphi)$, $R_u(t,\varphi,\rho) = \hat{R}(t,\varphi,\rho)$; thus the uniqueness is proved. Q.E.D.

In the sequel, it will be helpful to separate out the u-dependent terms in the coefficients of P_u, Q_u and R_u in (12b, c, d) and to eliminate the u-dependence of the kernels K_u and $\tilde{K}_u$ in (10). Write (12b, c, d) as

$$\frac{dP_u(t)}{dt} + A'(t)P_u(t) + P_u(t)A(t) + Q_u(t,0) + Q_u'(t,0) = -\hat{M}_u(t) \tag{12b'}$$

$$\begin{aligned} 2P_u(t)C(t,\varphi) + A'(t)Q_u(t,\varphi) + Q_u'(t,\varphi)A(t) + 2\frac{\partial Q_u(t,\varphi)}{\partial t} - 2\frac{\partial Q_u(t,\varphi)}{\partial \varphi} \\ + R_u(t,\varphi,0) + R_u(t,0,\varphi) = -2\hat{L}_u(t,\varphi) \end{aligned} \tag{12c'}$$

$$\begin{aligned} C'(t,\varphi)Q_u(t,\rho) + Q'(t,\varphi)C(t,\rho) + \frac{\partial R(t,\varphi,\rho)}{\partial t} - \frac{\partial R(t,\varphi,\rho)}{\partial \varphi} \\ - \frac{\partial R(t,\varphi,\rho)}{\partial \rho} = -\hat{G}_u(t,\varphi,\rho), \end{aligned} \tag{12d'}$$

where

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† In fact, under the weaker hypothesis of the last footnote, the equalities hold between Q_u, $\hat{Q}$ and R_u, $\hat{R}$ almost everywhere in (φ,ρ) for each t.

$$(16a)\qquad \hat{M}_u(t) = M_u(t) + E_u'(t)D'(t)P_u(t) + P_u(t)D(t)E_u(t)$$

$$(16b)\qquad \hat{L}_u(t,\varphi) = L_u(t,\varphi) + P_u(t)D(t)F_u(t,\varphi) + \tfrac{1}{2}[E_u'(t)D'(t)Q_u(t,\varphi) + Q_u'(t,\varphi)D(t)E_u(t)]$$

$$(16c)\qquad \hat{G}_u(t,\varphi,\rho) = G_u(t,\varphi,\rho) + F_u'(t,\varphi)D'(t)Q_u(t,\rho) + Q_u'(t,\varphi)D(t)F_u(t,\rho).$$

The boundary conditions (13a, e) do not depend on u.

<u>Theorem 2.</u> <u>Suppose the conditions of Theorem 1. Define</u> $\hat{P}_{ui}$, $\hat{Q}_{ui}$ <u>and</u> $\hat{R}_{ui}$, <u>as the terms in</u> (9', 10', 11'), <u>or equivalently, the respective terms in</u> (9) - (11) <u>with</u> K, $\tilde{K}$, $\hat{M}_u$, $\hat{L}_u$ <u>and</u> $\hat{G}_u$ <u>replacing</u> K_u, $\tilde{K}_u$, M_u, L_u <u>and</u> G_u, <u>resp.</u> <u>Then</u>

$$(17)\qquad P_u(t) = \sum_1^4 \hat{P}_{ui}(t),\quad Q_u(t,\varphi) = \sum_1^4 \hat{Q}_{ui}(t,\varphi),\ R_u(t,\varphi,\rho) = \sum_1^4 \hat{R}_{ui}(t,\varphi,\rho)$$

$$(9a')\qquad \hat{P}_{u1}(t) = \int_t^T K'(s,t)\hat{M}_u(s)K(s,t)ds$$

$$(9b')\qquad \hat{P}_{u2}(t) = \int_t^T ds \int_{-r}^0 d\tau K'(s,t)\hat{L}_u(s,\tau)K(s+\tau,t)$$

$$(9c')\qquad \hat{P}_{u3}(t) = P_{u2}'(t)$$

$$(9d')\qquad \hat{P}_{u4}(t) = \int_t^T ds \int_{-r}^0 d\varphi \int_{-r}^0 d\rho K'(s+\varphi,t)\hat{G}_u(s,\varphi,\rho)K(s+\rho,t)$$

$$(10a')\qquad \hat{Q}_{u1}(t,\varphi) = \int_t^T ds K'(s,t)\hat{M}_u(s)\tilde{K}(s,t,\varphi)$$

$$(10b')\qquad \hat{Q}_{u2}(t,\varphi) = \int_t^T ds \int_{-r}^0 d\tau K'(s,t)\hat{L}_u(s,\tau)\tilde{K}(s+\tau,t,\varphi) + \int_t^{\min[t+r+\varphi,T]} K'(s,t)\hat{L}_u(s,t-s+\varphi)ds$$

$$\hat{Q}_{u3}(t,\varphi) = \int_t^T ds \int_{-r}^{0} d\tau K'(s+\tau,t)\hat{L}_u'(s,\tau)\tilde{K}(s,t,\varphi) \tag{10c'}$$

$$\begin{aligned}\hat{Q}_{u4}(t,\varphi) &= \int_t^T ds \int_{-r}^{0} d\alpha \int_{-r}^{0} d\rho K'(s+\alpha,t)\hat{G}_u(s,\alpha,\rho)K(s+\rho,t,\varphi)\\ &+ \int_t^{\min[t+\varphi+r,T]} ds \int_{-r}^{0} d\alpha K'(s+\alpha,t)\hat{G}_u(s,\alpha,t-s+\varphi)\end{aligned} \tag{10d'}$$

$$\hat{R}_{u1}(t,\varphi,\rho) = \int_t^T ds\tilde{K}'(s,t,\varphi)\hat{M}_u(s)\tilde{K}(s,t,\rho) \tag{11a'}$$

$$\begin{aligned}\hat{R}_{u2}(t,\varphi,\rho) &= \int_t^T ds \int_{-r}^{0} d\tau\tilde{K}'(s,t,\varphi)\hat{L}_u(s,\tau)\tilde{K}(s+\tau,t,\rho)\\ &+ \int_t^{\min[t+r+\rho,T]} \tilde{K}(s,t,\varphi)\hat{L}_u(s,t-s+\rho)ds\end{aligned} \tag{11b'}$$

$$\hat{R}_{u3}(t,\varphi,\rho) = R_{u2}'(t,\rho,\varphi) \tag{11c'}$$

$$\begin{aligned}\hat{R}_{u4}(t,\varphi,\rho) &= \int_t^T ds \int_{-r}^{0} d\alpha \int_{-r}^{0} d\beta\tilde{K}'(s+\alpha,t,\varphi)\hat{G}_u(s,\alpha,\beta)\tilde{K}(s+\beta,t,\rho)\\ &+ \int_t^{\min[t+r+\varphi,t+r+\rho,T]} \hat{G}_u(s,t-s+\varphi,t-s+\rho)ds\\ &+ \int_t^{\min[t+r+\rho,T]} ds \int_{-r}^{0} d\alpha\tilde{K}'(s+\alpha,t,\varphi)\hat{G}_u(s,\alpha,t-s+\rho)\\ &+ \int_t^{\min[t+r+\varphi,T]} ds \int_{-r}^{0} d\alpha\hat{G}_u(s,\alpha,t-s+\varphi)\tilde{K}(s+\alpha,t,\rho).\end{aligned} \tag{11d'}$$

<u>Proof</u>. In the integrals (9) in the expression $\sum_1^4 P_{ui}(t)$, replace K_u and $\tilde{K}_u$ by K and $\tilde{K}$, resp., and M_u, L_u, G_u by $\hat{M}_u$, $\hat{L}_u$, $\hat{G}_u$, resp.. In Theorem 1, let $u \equiv 0$, $L_o = \hat{L}_u$, $M_o = \hat{M}_u$, $G_o = \hat{G}_u$. With this replacement, the P_{ui} terms in (9) become the $\hat{P}_{ui}$ terms in (9'). Then, by Theorem 1, the $\hat{P}_{ui}(t)$ are differentiable, and $\sum_1^4 \hat{P}_{ui}(t) = \hat{P}_u(t)$ satisfies (12b')(or equivalently, (12b)). Similarly for $\sum_1^4 \hat{Q}_{ui}(t,\varphi) = \hat{Q}_u(t,\varphi)$ and

$\sum_1^4 \hat{R}_{ui}(t,\varphi,\rho) = \hat{R}_u(t,\varphi,\rho)$. Then, by the symmetry of $\hat{P}_u(t)$ and $\hat{R}_u(t,\varphi,\rho)$ and the uniqueness part of Theorem 1, we have (17). Q.E.D.

Theorem 3. Suppose that $N(t)$, $M(t)$, $A(t)$, $B(t)$, $C(t,\varphi)$, $D(t)$, and $E_u(t)$ and $F_u(t,\varphi)$ satisfy the conditions of Theorem 1. Then the $P_{ui}(t)$, $Q_{ui}(t,\varphi)$ and $R_{ui}(t,\varphi,\rho)$ of (9) - (11) are continuously differentiable in their arguments for $0 \le t \le T$, $-r \le \varphi \le 0$, $-r \le \rho \le 0$, except that the φ or ρ derivatives of $R_{u2}(t,\varphi,\rho)$ and $R_{u3}(t,\varphi,\rho)$ may be discontinuous at $\varphi = \rho$. However, $R_u(t,\varphi,\rho)$ has a derivative in the $(1,-1,-1)$ direction.

Proof. Since the evaluations are tedious and straightforward, we give the details for one 'typical' term only, namely $Q_{u2}(t,\varphi)$. We note only that the asserted discontinuity in R_{u2} arises from the latter term of (11b') and that it is easy to verify that $(\partial/\partial t - \partial/\partial\rho)$ applied to this latter term yields a continuous function. For future reference note that the discontinuity is uniformly bounded if the L_u are. Write

$$Q_{u2}(t,\varphi) = \int_t^T \int_{-r}^{o} K_u'(s,t)L_u(s,\tau)\tilde{K}_u(s+\tau,t,\varphi)\,ds\,d\tau$$
$$+ \int_t^{\min[t+r+\varphi,T]} K_u'(s,t)L_u(s,t-s+\varphi)\,ds.$$

Recall that $L_u(t,\varphi) = E_u'(t)N(t)F_u(t,\varphi)$.

Denote the second term of $Q_{u2}(t,\varphi)$ by $\beta(t,\varphi)$. Observe that it is continuous in (t,φ). Let $t + r + \varphi > T$. Then

$$\partial\beta(t,\varphi)/\partial\rho = \int_t^T K_u'(s,t)\frac{\partial L_u}{\partial\rho}(s,t-s+\varphi)\,ds$$

which is continuous in (t,φ). For $t + r + \varphi < T$, we have

$$\partial\beta(t,\varphi)/\partial\varphi = K_u'(t+r+\varphi,t)L_u(t+r+\varphi,-r) + \int_t^T K_u'(s,t)\frac{\partial L_u}{\partial\varphi}(s,t-s+\varphi)$$

which is continuous in (t,φ) in the desired range. In addition, $L_u(t+r+\varphi,-r) \to 0$ as $t + r + \varphi \to T$, since $F_u(t,\varphi) \to 0$ as $t \to T$. Thus $\beta(t,\varphi)$ has continuous φ derivatives for $t,\varphi \in [0,T] \times [-r,0]$. The details for $\partial\beta(t,\varphi)/\partial t$ are similar and are omitted.

Write the first term of $Q_{u2}(t,\varphi)$ as

$$\alpha(t,\varphi) = \int_t^T h(s,\varphi,t)ds$$

where

$$h(s,\varphi,t) = \int_{\max(t-s+\varphi,-r)}^{0} K_u'(s,t)L_u(s,\tau)\tilde{K}_u(s+\tau,t,\varphi)ds.$$

If $t - s + \varphi > 0$, the lower limit is replaced by zero.

For each fixed $t \geq 0$ let $k(s,\varphi,t)$ satisfy (a): $k(s,\varphi,t)$ is continuous on $[t,T] \times [-r,0]$; (b): There is a bounded measurable function $k_\varphi(s,\varphi,t)$ so that for each t and each s - not in some null set in $[t,T]$, $k_\varphi(s,\varphi,t) = \partial k(s,\varphi,t)/\partial\varphi$ for almost all φ in $[-r,0]$; (c): $\int_t^T k_\varphi(s,\varphi,t)ds$ is continuous on $[0,T] \times [-r,0]$. Then $\int_t^T k_\varphi(s,\varphi,t)ds = \partial/\partial\varphi\int_t^T k(s,\varphi,t)ds$ and is continuous on $[0,T] \times [-r,0]$. Let $k(s,\varphi,t) = h(s,\varphi,t)$, and note that $h(s,\varphi,t)$ is continuous for each fixed t. Let $t - s + \varphi < -r$. Then $\delta_1(s,\varphi,t) \equiv \partial h(s,\varphi,t)/\partial\varphi = \int_{-r}^{0} K_u'(s,t)L_u(s,\tau)\frac{\partial\tilde{K}_u}{\partial\varphi}(s+\tau,t,\varphi)ds$ which is continuous in all three variables.

Now, let $0 > t - s + \varphi > -r$. Then

$$\delta_2(s,\varphi,t) = \frac{\partial k(s,\varphi,t)}{\partial\varphi} = K_u'(s,t)L_u(s,t-s+\varphi)\tilde{K}_u(t+\varphi,t,\varphi)$$
$$+ \int_{t-s+\varphi}^{0} K_u'(s,t)L_u(s,\tau)\frac{\partial\tilde{K}_u}{\partial\varphi}(s+\tau,t,\varphi)d\tau.$$

The first term of $\delta_2(s,\varphi,t)$ is zero since $\tilde{K}_u(t+\varphi,t,\varphi) = 0$ and the second tends to $\delta_1(s,\varphi,t)$ as $t - s + \varphi \downarrow -r$. It can now easily be verified that (a) - (c) hold and that $\alpha(t,\varphi)$ has a continuous φ derivative on $[0,T] \times [-r,0]$. The details for $\partial\alpha(t,\varphi)/\partial t$ are similar and are omitted. Q.E.D.

4. Iteration in Policy Space.

In Theorem 4, the basic result on 'iteration in policy space', we will require the time derivative of the function $V^u(x_t,t)$ evaluated on the path corresponding to a control w (and written $\dot{V}^{u,w}(x_t,t)$); to be specific, the time derivative of $V^u(x_t,t)$ along the path corresponding to w is defined by

$$\dot{V}^{u,w}(x_t,t) = \frac{\partial}{\partial t}[x'(t)P_u(t)x(t) + 2x'(t)\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi$$
$$(18) \qquad + \int_{-r}^{0}\int_{-r}^{0} x'(t+\varphi)R_u(t,\varphi,\rho)x(t+\rho)d\rho d\varphi]$$

where for $\dot{x}(t) \equiv \partial x(t)/\partial t$ we use the derivative evaluated along the trajectory corresponding to w; i.e.,

$$(19) \qquad \dot{x}(t) = A(t)x(t) + B(t)x(t-r) + D(t)w(t) + \int_{-r}^{0} C(t,\varphi)x(t+\varphi)d\varphi.$$

Using (19) in the calculations (13), we have

$$\dot{V}^{u,w}(x_t,t) = 2w'(t)D(t)P_u(t)x(t) + 2w'(t)D'(t)\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi$$

$$+ x'(t)\left[\frac{dP_u(t)}{dt} + A'(t)P_u(t) + P_u(t)A(t) + Q_u(t,0) + Q_u'(t,0)\right]x(t)$$

$$+ x'(t)\int_{-r}^{0}[2(\partial/\partial t-\partial/\partial\varphi)Q_u(t,\varphi) + 2P_u(t)C(t,\varphi)$$

(19a)
$$+ A'(t)Q_u(t,\varphi) + Q_u'(t,\varphi)A(t)$$

$$+ R_u(t,\varphi,0) + R_u(t,0,\varphi)]x(t+\varphi)d\varphi$$

$$+ \int_{-r}^{0}\int_{-r}^{0} x'(t+\varphi)[(\partial/\partial t-\partial/\partial\varphi-\partial/\partial\rho)R_u(t,\varphi,\rho)$$

$$+ C'(t,\varphi)Q_u(t,\rho) + Q'(t,\varphi)C(t,\rho)]x(t+\rho)d\varphi d\rho.$$

<u>Theorem 4</u>. <u>Let</u> u <u>have the form</u> (5), <u>and define</u> $\dot{V}^{u,w}(x_t,t)$ <u>by</u> (18). <u>Assume the conditions on</u> A, B, C, D, E_u, F_u, N <u>and</u> M <u>of Theorem</u> 1, <u>and let</u> $N(s)$ <u>be positive definite and</u> $M(s)$ <u>positive semi-definite in</u> $[0,T]$, <u>and let</u> $D(t)$ <u>be continuously differentiable in</u> $[0,T]$. <u>The control</u> w <u>which attains the minimum in</u> (22) <u>has the form</u> (5), <u>and</u>

(20a)
$$w(t) = E_w(t)x(t) + \int_{-r}^{0} F_w(t,\varphi)x(t+\varphi)d\varphi$$

<u>where</u>

$$E_w(t) = -N^{-1}(t)D'(t)P_u(t)$$

$$F_w(t,\varphi) = -N^{-1}(t)D'(t)Q_u(t,\varphi).$$

$E_w(t)$ <u>and</u> $F_w(t,\varphi)$ <u>satisfy the conditions on the</u> $E_u(t)$ <u>and</u> $F_u(t,\varphi)$ <u>in</u>

<u>Theorem 1.</u> <u>Also</u>

(21) $$V^w(x_t,t) \leq V^u(x_t,t)$$

<u>for all</u> $x_t \in H$, <u>and</u> $t \in [0,T]$.

(22) $$H(x_t,t) = \min_w [\dot{V}^{u,w}(x_t,t) + x'(t)M(t)x(t) + w'(t)N(t)w(t)].$$

<u>Remark.</u> Note that, with $w = u$, the bracketed term in (22) is zero by the definition of $\dot{V}^{u,u}(x_t,t) = \partial V^u(x_t,t)/\partial t$.

<u>Proof.</u> In computing the minimum in (22), only the terms

(23a) $$\dot{x}'(t)P_u(t)x(t) + x'(t)P_u(t)\dot{x}(t) + 2\dot{x}'(t)\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi$$
$$+ w'(t)N(t)w(t)$$

or, equivalently, only the terms

(23b) $$2w'(t)D'(t)P_u(t)x(t) + 2w'(t)D'(t)\int_{-r}^{0} Q_u(t,\varphi)x(t+\varphi)d\varphi$$
$$+ w'(t)N(t)w(t)$$

need be taken into account. The other terms in the brackets in (22) do not contain w by (19a). The $w(t)$ minimizing (23b) is of the form (20a), where E_w and F_w satisfy (20b). By the hypothesis and by Theorem 1, the coefficients E_w and F_w satisfy the smoothness conditions required in Theorem 1 on the E_u, F_u there.

Now, for any w of the form (20), $V^u(x_T,T) = V^w(x_T,T) = 0$ and

$$\int_t^T \dot{V}^{u,w}(x_t,t) = V^u(x_T,T) - V^u(x_t,t).$$

The bracketed term in (22), with the minimizing w inserted, is non-positive - since the bracketed term is zero if w is replaced by u. Thus

$$0 \geq \int_t^T \dot{V}^{u,v}(x_s,s)ds + \int_t^T [x'(s)M(s)x(s) + w'(s)N(s)w(s)]ds$$

or

$$0 \geq V^u(x_T,T) - V^u(x_t,t) + V^w(x_t,t) - V^w(x_T,T) = -V^u(x_t,t) + V^w(x_t,t)$$

and (21) holds. Q.E.D.

Suppose the conditions on A, B, C, D, N and M of Theorem 4. Let u_o satisfy the conditions in the remark below Lemma 1. Define the improved control u_n recursively in terms of u_{n-1} by the method of Theorem 4. Then, by Theorem 4 (where we write $E_n = E_{u_n}$, $F_n = F_{u_n}$, $V^n = V^{u_n}$)

$$u_n = E_n(t)x(t) + \int_{-r}^{0} F_n(t,\varphi)x(t+\varphi)d\varphi \tag{24}$$

$$\begin{aligned} E_{n+1}(t) &= -N^{-1}(t)D'(t)P_n(t) \\ F_{n+1}(t,\varphi) &= -N^{-1}(t)D'(t)Q_n(t,\varphi) \end{aligned} \tag{25}$$

and, for all $t \in [0,T]$ and $x_t \in H$,

$$V^{n+1}(x_t,t) \le V^n(x_t,t). \tag{26}$$

Next, it is shown that (26) implies that the P_n, Q_n, R_n and u_n converge.

Theorem 5. Assume the conditions of Theorem 4. The $P_n(t)$, $Q_n(t,\varphi)$, $R_n(t,\varphi,\rho)$, $E_n(t)$, *and* $F_n(t,\varphi)$ *are uniformly bounded and converge pointwise to functions* $P(t)$, $Q(t,\varphi)$, $R(t,\varphi,\rho)$, $E(t)$ *and* $F(t,\varphi)$, *resp.* $P(t)$ *and* $R(t,\varphi,\rho)$ *are symmetric and*

$$\begin{aligned} V^u(x_t,t) = {} & x'(t)P(t)x(t) + x'(t)\int_{-r}^{0} Q(t,\varphi)x(t+\varphi)d\varphi \\ & + \int_{-r}^{0} x'(t+\varphi)Q'(t,\varphi)x(t)d\varphi \\ & + \int_{-r}^{0}\int_{-r}^{0} x'(t+\varphi)R(t,\varphi,\rho)x(t+\rho)d\varphi d\rho \end{aligned} \tag{27}$$

where u *is the limit of the* u_n:

$$u(t) = E(t)x(t) + \int_{-r}^{0} F(t,\varphi)x(t+\varphi)d\varphi. \tag{28}$$

Furthermore, the $\hat{M}_n$, $\hat{G}_n$ and $\hat{L}_n$ in (9') (11') converge pointwise and are uniformly bounded, and the P, Q and R are the limits of the sums of the P_{ni}, Q_{ni} and R_{ni}, resp.

Finally, let v be the $(1,-1)$ direction in the (t,φ) set $[0,T] \times [-r,0]$, and σ the $(1,-1,-1)$ direction in the (t,φ,ρ) set $[0,T] \times [-r,0]^2$. Then the derivatives $\partial P(t)/\partial t$, $\partial Q(t,\varphi)/\partial v$, $\partial R(t,\varphi,\rho)/\partial\sigma$ exist and satisfy

$$\frac{\partial P(t)}{\partial t} + A'(t)P(t) + P(t)A(t) + Q(t,0) + Q'(t,0) = -\hat{M}(t) \tag{29a}$$

$$\begin{aligned} 2\sqrt{2}\,\frac{\partial Q(t,\varphi)}{\partial v} &+ 2P(t)C(t,\varphi) + A'(t)Q(t,\varphi) + Q'(t,\varphi)A(t) \\ &+ R(t,\varphi,0) + R(t,0,\varphi) = -2\hat{L}(t,\varphi) \end{aligned} \tag{29b}$$

$$\sqrt{3}\,\frac{\partial R(t,\varphi,\rho)}{\partial\sigma} + C'(t,\varphi)Q(t,\rho) + Q'(t,\varphi)C(t,\rho) = -\hat{G}(t,\varphi,\rho) \tag{29c}$$

where the $\hat{M}$, $\hat{L}$ and $\hat{G}$ are the $\hat{M}_n$, $\hat{G}_n$, $\hat{L}_n$, with E_n and F_n replaced by their limit. Also

$$\begin{aligned} &B'(t)P(t) - Q(t,-r) = 0 \\ &B'(t)Q(t,\varphi) - R(t,-r,\varphi) - R'(t,\varphi,-r) \\ &\qquad + Q'(t,\varphi)B(t) = 0. \end{aligned} \tag{29d}$$

$\partial P_n(t)/\partial t$, $\partial Q_n(t,\varphi)/\partial v$ and $\partial R_n(t,\varphi,\rho)/\partial\sigma$ converge to $\partial P(t)/\partial t$, $\partial Q(t,\varphi)/\partial v$ and $\partial R(t,\varphi,\rho)/\partial\sigma$, resp.

Proof. The other statements follow readily from the uniform boundedness and convergence of the P_n, Q_n and R_n and Theorems 1 and 2; hence only this will be shown.

We note only that $(\partial/\partial t-\partial/\partial\varphi)Q_n(t,\varphi) = \sqrt{2}\,\partial Q_n(t,\varphi)/\partial v$, and $(\partial/\partial t-\partial/\partial\varphi-\partial/\partial\rho)R_n(t,\varphi,\rho) = \sqrt{3}\,\partial R_n(t,\varphi,\rho)/\partial\sigma$. These derivatives converge if the P_n, Q_n, and R_n do, and are uniformly bounded by (12) and (12'). If the P_n, Q_n and R_n and their $(t,v,\sigma$, resp.) derivatives all converge then the $(t,r,\sigma$, resp.) derivatives of the limits are the limits of the $(t,r,\sigma$, resp.) derivatives. In (26), let $x(t+\varphi) = 0$ for $\varphi \neq 0$. Then (26) implies that $x'P_{n+1}(t)x \leq x'P_n(t)x$ for any vector x. Hence, $P_n(t)$ converges pointwise to a symmetric measurable matrix $P(t)$. Since the diagonal elements $p_{n,ii}(t)$ are non-increasing, and $|p_{n,ij}(t)| \leq \max_i p_{n,ii}(t)$, the $P_n(t)$ are uniformly bounded.

Let $x(\varphi)$ be any continuous function on $[-r,0]$ with $x(0) = 0$. Then, for such $x(\varphi)$, (26) implies that

$$(30)\qquad \int_{-r}^{0}\int_{-r}^{0} x'(\varphi)R_{n+1}(t,\varphi,\rho)x(\rho)d\varphi d\rho \leq \int_{-\sigma}^{0}\int_{-\sigma}^{0} x'(\varphi)R_n(t,\varphi,\rho)x(\rho)d\varphi d\rho.$$

By the continuity of the $R_n(t,\varphi,\rho)$, (30) holds if $x(\varphi)$ is a Dirac δ-function. In particular, if $-r < \varphi_o < 0$, $-r < \rho_o < 0$ and $x(\varphi) = x\delta(\varphi-\varphi_o) + y\delta(\varphi-\rho_o)$, then (30) and the fact that $R_n'(t,\varphi,\rho) = R_n(t,\rho,\varphi)$ yields

$$(31)\qquad \begin{aligned} &x'R_{n+1}(t,\varphi_o,\varphi_o)x + y'R_{n+1}(t,\rho_o,\rho_o)y + 2x'R_{n+1}(t,\varphi_o,\rho_o)y \\ &\qquad \leq x'R_n(t,\varphi_o,\varphi_o)x + y'R_n(t,\rho_o,\rho_o)y + 2x'R_n(t,\varphi_o,\rho_o)y \end{aligned}$$

But, by continuity of the $R_n(t,\varphi,\rho)$, (31) holds for any φ_o, ρ_o in $[-r,0]$. Let $y = 0$. Then, as shown for the P_n, (31) implies that the $R_n(t,\varphi,\varphi)$ are uniformly bounded and converge to some $R(t,\varphi,\varphi)$. Using this and (31) and the arbitrariness of x,y implies that the $R_n(t,\varphi,\rho)$ are uniformly bounded and that $R_n(t,\varphi,\rho)$ converges to some $R(t,\varphi,\rho)$. By similar reasoning, (26) implies that, for each $\varphi_o \in [-r,0]$,

$$
\begin{aligned}
&x'P_{n+1}(t)x + 2x'Q_{n+1}(t,\varphi_o)y + y'R_{n+1}(t,\varphi_o,\varphi_o)y \\
&\qquad \leq x'P_n(t)x + 2x'Q_n(t,\varphi_o)y + y'R_n(t,\varphi_o,\varphi_o)y.
\end{aligned}
\tag{32}
$$

Using (32) and the conclusions concerning P_n and R_n, we may deduce that the $Q_{n+1}(t,\varphi)$ converges to some $Q(t,\varphi)$ and are uniformly bounded. Q.E.D.

Corollary. For any control $w(t)$ which gives bounded continuous paths $x(t)$, and which is bounded for any bounded continuous initial condition $\dot{V}^{u,w}(x_t,t)$ exists and $\dot{V}^{u_n,w}(x_t,t)$ converges to it for any continuous initial condition. The class of $w(t)$ includes all controls which are linear in x_t and have bounded coefficients.

Note. Recall that $\dot{V}^{u,w}(x_t,t)$ is the time derivative of $V^u(x_t,t)$ along x_t paths corresponding to the control w.

Proof. Since $V^{u_n}(x_t,t)$ converges to $V^u(x_t,t)$ for any continuous initial condition, we only need to show that $\dot{V}^{u_n,w}(x_t,t)$ is uniformly bounded (in n) and converges for any continuous initial condition. $\dot{V}^{u_n,w}(x_t,t)$ is given by (19a) with u_n replacing u_n, and Theorem 5 implies that $\dot{V}^{u_n,w}(x_t,t)$ converges. Q.E.D.

5. The Optimality Theorem.

Theorem 6. Let $w(x,t)$ be any control for which a solution to (1) is defined on $[0,T]$ for any initial condition, and let u be given by (28). Then $V^u(x_t,t) \leq V^w(x_t,t)$ for all t, and initial conditions x_t. Let $u = w$ and E_u and F_u be given by (28). Then the set of equations (29) has a unique solution (for symmetric $P(t)$ and $R(t,\varphi,\rho)$) and determines the optimal control w.

Proof. Calculating the minimizing w in (32) (see Theorem 4 for terminology)

$$\min_{w} [\dot{V}^{u,w}(x_t,t) + x'(t)M(t)x(t) + w'(t)N(t)w(t)] \tag{32}$$

yields (see (19a))

$$w(x_t,t) = -N^{-1}(t)D'(t)[P(t)x(t) + \int_{-r}^{0} Q(t,\varphi)x(t+\varphi)d\varphi],$$

which is exactly u. Also the bracketed term in (32) is zero if u replaces w. Thus, for any $u \neq w$, we have

$$\dot{V}^{u,w}(x_t,t) + x'(t)M(t)x(t) + w'(t)N(t)w(t) \geq 0$$

or

$$0 \leq \int_t^T \dot{V}^{u,w}(x_s,s)ds + \int_t^T [x'(s)M(s)x(s) + w'(s)N(s)w(s)]ds$$

$$= -V^u(x_t,t) + V^u(x_T,T) + V^w(x_t,t) - V^w(x_T,T)$$

or, equivalently, $V^{w}(x_t,t) \geq V^{u}(x_t,t)$. The last sentence of the Theorem follows from Theorems 5 and 2. Q.E.D.

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